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Global Minimization of a Multivariate Polynomial using Matrix Methods

BERNARD HANZON[1] and DORINA JIBETEAN[2]
[1]*Vrije Universiteit, De Boelelaan 1105, 1081 HV Amsterdam, The Netherlands. (e-mail: bhanzon@econ.vu.nl);* [2]*Centrum voor Wiskunde en Informatica, P.O. Box 94079, 1090 GB Amsterdam, The Netherlands. (e-mail: d.jibetean@cwi.nl)*

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Abstract. The problem of minimizing a polynomial function in several variables over $\mathbf{R}^n$ is considered and an algorithm is given. When the polynomial has a minimum the algorithm returns the global minimal value and finds at least one point in every connected component of the set of minimizers. A characterization of such points is given. When the polynomial does not have a minimum the algorithm computes its infimum. No assumption is made on the polynomial.

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1. Introduction

Although optimization problems have been treated extensively, the particular case of polynomial optimization has not received much attention. We will mention here some approaches to the problem of (constrained) polynomial optimization we have encountered. Exact methods can be found in Hägglöf et al. (1995) and Uteshev and Cherkasov (1998). The first paper looks at the first order conditions. They form a system of polynomial equations which can be solved for example by using Gröbner basis techniques. However, in the case of an infinite number of critical points, even when a Gröbner basis can be computed, its elements may describe very complicated sets of points. It is not clear how one would proceed from there. The second paper mentioned makes some assumptions on the given polynomial restricting in this way its applicability.

The algorithms mentioned above work when the given polynomial has a minimum, without considering an approach for finding the infimum.

Different approaches, based on solving a certain convex relaxation of the problem, can be found in Shor (1998), Parrilo (2001) or Lassere (2001). Such methods seem to have better computational properties. However, in general, they only guaranty finding a lower bound of the infimum.

Let us now state the problem more precisely. Let $p \in \mathbf{R}[x_1, x_2, ..., x_n]$ be a polynomial of total degree larger than 1. In this paper we give an algorithmic solution to the problem of minimizing p over $\mathbf{R}^n$. The algorithm is guaranteed to find the minimal value of p, when this exists. When the polynomial has a finite number of points where the minimum is attained, the algorithm finds all of them. In case the number of points of (global) minimum is infinite, there is still a finite number of connected components composing the set $p^{-1}(\{\min_{x\in R^n} p(x)\})$. The algorithm will return at least one point in every connected component. In case the polynomial has a finite infimum, the algorithm will return this value. We should remark here that a multivariate polynomial may have a finite infimum, as one may see from the following example considered in Uteshev and Cherkasov (1998): $p(x_1, x_2) = x_1^2 x_2^4 + x_1 x_2^2 + x_1^2$ for which $\inf_{(x_1,x_2)\in\mathbf{R}^2} p(x_1, x_2) = -1/4$. However the infimum is reached at infinity. Therefore, one has to distinguish between the cases when p has a minimum, a finite infimum or an infinite infimum.

In this paper, no assumptions are made on the polynomial p. Note also that we do not include in this setting any domain constraints.

The remainder of the paper is organized as follows. Section 2 introduces few notions used for systems of polynomial equations and describes the Stetter-Möller method for finding all critical points of a polynomial when the number of critical points is finite. In Section 3 we propose a certain perturbation on the original problem which allows us to treat, using the same method, the case of an infinite number of critical points and we give some theoretical results. Section 4 deals with the actual computations, describing in more detail the output of the algorithm. In the end, in Sections 5 and 6, we discuss the algorithm in two particular cases and draw the conclusions.

2. Solving polynomial equations

When minimizing a function in n variables one often looks at the first order conditions. They form a system of (nonlinear) equations in n variables. Systems of polynomial equations received much attention and methods like Gröbner bases calculation and Stetter-Möller method were proposed for solving them. We assume that the reader is familiar with the theory of Gröbner bases but we discuss the second mentioned method.

2.1. PRELIMINARY NOTIONS AND NOTATION

To begin, we recall some definitions and results regarding the solution set of a system of polynomial equations. Let K be a field. Given a set of polynomials $f_1, \ldots, f_s \in K[x_1, \ldots, x_n]$ one defines

$$< f_1, \ldots, f_s >= \{p_1 f_1 + \ldots + p_s f_s \ : \ p_i \in K[x_1, \ldots, x_n], \ i = 1, \ldots, s\}$$

to be the ideal generated by $f_1, \dots, f_s$. It can be shown easily that the set $< f_1, \dots, f_s >$ is indeed an ideal. The set of all simultaneous solutions in K^n of a system of equations

$$\{(x_1, \dots, x_n) \in K^n \mid f_1(x_1, \dots, x_n) = 0, \\ f_2(x_1, \dots, x_n) = 0, \dots, \; f_s(x_1, \dots, x_n) = 0\}$$

is called the affine variety (or algebraic set) defined by $f_1, \dots, f_s$ and it is denoted by $V(f_1, \dots, f_s)$. Given a polynomial ideal I one can define the associated affine variety

$$V(I) = \{(x_1, \dots, x_n) \in K^n \mid \forall f \in I, \; f(x_1, \dots, x_n) = 0\}.$$

Note that if $I =< f_1, \dots, f_s >$, then $V(I)$ is also denoted $V(f_1, \dots, f_s)$.

The generating sets of polynomials for a polynomial ideal I play an important role. Obviously, any finite set of polynomials generates a polynomial ideal. The converse is also true: given a polynomial ideal, there exists a finite set of polynomials which generates it. Note however that the generating set is not unique. The uniqueness of a basis can be obtained by imposing some supplementary conditions on its elements. In fact, any nontrivial ideal has a unique monic *reduced Gröbner basis* associated to a given monomial ordering (see Cox et al. 1998, Ch. 1 §3). There are algorithms for computing such a Gröbner basis but in general they have high computational complexity (see Geddes et al., 1992). More details about Gröbner bases and their properties can be found for example in Cox et al. (1997).

2.2. STETTER-MÖLLER METHOD FOR SOLVING SYSTEMS OF POLYNOMIAL EQUATIONS

This section is based on Cox et al. (1998), Hanzon et al. (1998) and Stetter (1996). Given a polynomial ideal I one can define the quotient space $K[x_1, \dots, x_n]/I$. This set together with an internal addition operation and a scalar multiplication operation has a vector space structure. The elements of this space are classes of polynomials of the form $[f] = \hat{f} + I$. If $G = \{g_1, \dots, g_n\}$ is a Gröbner basis for I, then for every $f \in K[x_1, \dots, x_n]$ there exists a unique $\hat{f} \in K[x_1, \dots, x_n]$ such that $f = f_1 g_1 + \dots + f_n g_n + \hat{f}$ and no term of $\hat{f}$ is divisible by any of the leading terms of the elements in G. $\hat{f}$ is called the remainder of the division of f by G. Obviously, the remainder is zero if and only if $f \in I$ and polynomials in the same class have the same remainder. The following theorem (Finiteness Theorem of Cox et al., 1998), characterizing the finite dimensional quotient spaces, is of importance for us.

THEOREM 2.1. *Let $K \subseteq \mathbf{C}$ and $I \subseteq K[x_1, \dots, x_n]$ be an ideal. The following conditions are equivalent:*

a. The vector space $K[x_1, \dots, x_n]/I$ is finite dimensional over K.

b. The associated variety $V(I) \subseteq \mathbf{C}$ is a finite set.
c. If G is a Gröbner basis for I, then for each i, $1 \leqslant i \leqslant n$, there is an $m_i \geqslant 0$ such that $x_i^{m_i}$ is the leading term of g for some $g \in G$.

From now on we take the field K to be equal to the field of complex numbers $\mathbf{C}$. Next we recall the Stetter-Möller method for solving a system of polynomial equations or, in other words, for calculating the points of the variety associated to the generated ideal. When the system of equations has finitely many solutions, that is when $\mathbf{C}[x_1, \ldots, x_n]/I$ is a finite dimensional vector space over $\mathbf{C}$, the method evaluates an arbitrary polynomial at the points of $V(I)$. In particular, considering f equal to x_i, the method gives the coordinates of the points in $V(I)$. Let $f \in \mathbf{C}[x_1, \ldots, x_n]$ be an arbitrary polynomial. Define

$$A_f : \mathbf{C}[x_1, \ldots, x_n]/I \to \mathbf{C}[x_1, \ldots, x_n]/I, \quad A_f([g]) = [f][g] = [fg].$$

Note that the multiplication is well defined on $\mathbf{C}[x_1, \ldots, x_n]/I$ due to the fact that I is an ideal. As A_f is a linear mapping from a finite dimensional space to itself, there exists a matrix representation of it with respect to a basis of $\mathbf{C}[x_1, \ldots, x_n]/I$. Such a basis is given by the normal set associated to the reduced Gröbner basis, i.e. the set of monomials which are not divisible by any leading term of the Gröbner basis, $B = \{x^{\alpha(1)}, \ldots, x^{\alpha(m)}\}$. Here $\alpha(i) = (\alpha_1(i), \ldots, \alpha_n(i))$ and $x^{\alpha(i)} = \prod_{j=1}^n x_j^{\alpha_j(i)}$ for any $i = 1, \ldots, m$. Let N denote the cardinality of B. In the following we use the same notation for the linear mapping A_f as well as for the matrix associated to it. The following properties hold for the $N \times N$ matrices A_f.

PROPOSITION 2.2. *Let $f, g \in \mathbf{C}[x_1, \ldots, x_n]$. Then:*
a. $A_f = 0$ if and only if $f \in I$.
b. $A_{f+g} = A_f + A_g$.
c. $A_{fg} = A_f A_g$.
d. Given a polynomial $h \in \mathbf{C}[t]$ we have $A_{h(f)} = h(A_f)$

Consider the particular matrices A_{x_i}, $i = 1, \ldots, n$. Using the properties above it is not difficult to see that $(A_{x_1}, \ldots, A_{x_n})$ is in fact a matrix element in the variety $V(I)$, that is $\forall f \in I$, $f(A_{x_1}, \ldots, A_{x_n}) = 0$. Here 0 denotes the zero matrix and $f(A_{x_1}, \ldots, A_{x_n})$ is well-defined due to the commutativity of the matrices.

Since matrices $A_{x_1}, \ldots, A_{x_n}$ are pairwise commutative, they have common eigenvectors and the n-tuple $(\xi_1 \ldots, \xi_n)$ of eigenvalues of $A_{x_1}, \ldots, A_{x_n}$ respectively, corresponding to the same common eigenvector will be an element of $V(I)$. Moreover, *all* the points in $V(I)$ are found as n-tuples of eigenvalues corresponding to the same common eigenvector (see, e.g., Hanzon et al., 1998). For a general polynomial f we have:

THEOREM 2.3. *Let $I \subseteq \mathbf{C}[x_1, \ldots, x_n]$ be an ideal with the associated variety being zero-dimensional, $f \in \mathbf{C}[x_1, \ldots, x_n]$ and A_f the associated matrix. Then z is an eigenvalue of A_f if and only if z is a value of the function f on $V(I)$.*

In their papers, Möller and Stetter (1995), Stetter (1996) use instead of A_f the so-called *multiplication table* which is in fact the transpose of our matrix. By looking at the eigenvectors of the multiplication table (which in our case become the left eigenvectors) Stetter makes the interesting remark that if the eigenspace associated to a certain eigenvalue of A_f is 1-dimensional, then the vector $(\xi^{\alpha(1)}, \ldots, \xi^{\alpha(m)})$ is an eigenvector, where ξ is a solution of the system. In that case we call an eigenvector a Stetter vector. Hence, if $x_1, \ldots, x_n \in B$, the solutions of the system can be retrieved from the (left) eigenvectors of A_f.

3. Construction of an auxiliary polynomial

Consider a family of polynomials, depending on the *real positive* parameter λ, given by

$$q_\lambda(x_1, x_2, ..., x_n) = p(x_1, x_2, ..., x_n) + \lambda(x_1^{2m} + x_2^{2m} + \ldots + x_n^{2m}),$$

where m is a fixed positive integer with $m > \mathtt{tdeg}(p)/2$ and $\mathtt{tdeg}(p)$ stands for the *total degree of* p. One can rewrite $q_\lambda(x) = p(x) + \lambda\|x\|^{2m}$, where $\|x\|$ denotes the Minkowski $2m$ norm of $x = (x_1, x_2, ..., x_n)$.

If $\lambda > 0$ is fixed, the problem

$$\min_{x \in \mathbf{R}^n} q_\lambda(x_1, \ldots, x_n)$$

has two major advantages over the problem of finding $\inf_{x\in\mathbf{R}^n} p(x_1, \ldots, x_n)$. Firstly, the minimum of q_λ is always attained, hence the global minimum equals the smallest critical value. Secondly, the first order conditions, used to compute the critical points and critical values, form a reduced Gröbner basis with respect to any total degree ordering (irrespective of the ordering of the variables). Hence we can avoid computing a Gröbner basis by constructing one.

When λ goes to zero, from the family of polynomials q_λ we obtain again the polynomial p. We will study the relation between the minima of the polynomials q_λ and the infimum of p. Actually, we will prove that $\inf_{x\in R^n} p(x_1, x_2, ..., x_n) = \lim_{\lambda\downarrow 0} \min_{x\in R^n} q_\lambda(x_1, x_2, ..., x_n)$. Therefore, we can concentrate on solving the new problem $\min_{x\in R^n} q_\lambda(x_1, x_2, ..., x_n)$, from which we deduce the answer for the original one. But let us first discuss in detail the relation between the two problems.

PROPOSITION 3.1. *The first order conditions of the polynomial q_λ form a reduced Gröbner basis for the ideal generated by themselves.*

Proof. The partial derivatives of q_λ are $\partial q_\lambda(x)/\partial x_i = 2m\lambda x_i^{2m-1} + \partial p(x)/\partial x_i$, $\forall i = 1, \ldots, n$. With our choice of m, we have $2m > \mathtt{tdeg}(p)$ hence $2m - 1 > \mathtt{tdeg}(\partial p(x)/\partial x_i)$, $\forall i = 1, \ldots, n$. In other words, the leading term of $\partial q_\lambda(x)/\partial x_i$ is $2m\lambda x_i^{2m-1}$ and it depends on x_i only.

According to Cox et al. (1997 Ch. 2, § 9, Theorem 3 and Proposition 4), the set $\{\partial q_\lambda(x)/\partial x_i \mid i = 1, \ldots, n\}$ is a Gröbner basis (with respect to any total degree ordering). It is obvious that $\{\partial q_\lambda(x)/\partial x_i \mid i = 1, \dots, n\}$ is in fact a reduced Gröbner basis. □

Throughout the rest of the paper we use as a Gröbner basis, the set consisting of the derivatives of q_λ with respect to the variables $x_1, \dots, x_n$. The associated normal set B contains all monomials $\prod_{j=1}^n x_j^{\alpha_j}$ with α_j integer number, $0 \leqslant \alpha_j < 2m-1$. Therefore, the cardinality of B is $N = (2m-1)^n$.

In the following we discuss the relation between the infimum of the polynomial p and the minima of the polynomials q_λ.

LEMMA 3.2. *For every positive* λ, *the polynomial* q_λ *has a minimum.*

Proof. We want to show that for every $\lambda > 0$ there exists an r_λ such that the minimum of q_λ is reached inside the Minkowski ball $B(0, r_\lambda)$.

Let $x \in \mathbf{R}^n$ with the norm $\| x \| = r$. Then for every component of x we have $-r \leqslant x_i \leqslant r,\ i = 1, \dots, n$ and

$$q_\lambda(x) = \| x \|^{2m} \ (\lambda + p(x)/ \| x \|^{2m}).$$

But $-p_{abs}(r) \leqslant p(x) \leqslant p_{abs}(r)$ for all x with $\| x \| = r$ implies

$$r^{2m}(\lambda - p_{abs}(r)/r^{2m}) \ \leqslant \ q_\lambda(x).$$

Here p_{abs} is the polynomial obtained from p by replacing all its coefficients by their absolute value and taking all its variables equal. By construction we have that $2m$ is strictly larger than the total degree of the polynomial p (and also of p_{abs}), therefore $p_{abs}(r)/r^{2m}$ is a rational function in the variable r having the degree of the numerator strictly smaller than the degree of the denominator. Hence $\lim_{r\to\infty} p_{abs}(r)/r^{2m} = 0$ and so there exists an $r_\lambda^1 > 0$ such that for every $r \geqslant r_\lambda^1$ we have $\lambda > p_{abs}(r)/r^{2m}$. That means that for every x with $\| x \| = r \geqslant r_\lambda^1$ we have

$$0 \ < \ r^{2m}(\lambda - p_{abs}(r)/r^{2m}) \leqslant q_\lambda(x). \tag{1}$$

From (1) we see that $q_\lambda(x)$ goes to infinity for $r \to \infty,\ r = \|x\|$. Hence $\exists r_\lambda \geqslant r_\lambda^1$ such that $\forall r \geqslant r_\lambda$ and $x,\ \|x\| = r$, we have $q_\lambda(x) > q_\lambda(0)$, where $q_\lambda(0) = p(0)$ is a fixed number. Hence $\forall x,\ \|x\| \geqslant r_\lambda$ we have $q_\lambda(x) > q_\lambda(0)$ which implies that the minimum value of q_λ must be attained inside the Minkowski ball $B(0, r_\lambda)$. This completes our proof. □

Denote by X_λ the set of real points where the minimum of q_λ is attained

$$X_\lambda = \{x_\lambda \in \mathbf{R}^n \mid q_\lambda(x_\lambda) = \min_{x\in\mathbf{R}^n} q_\lambda(x)\}.$$

Elements of X_λ will be denoted by x_λ. From Lemma 3.2 we know that X_λ is nonempty for every λ positive. Also X_λ is a finite set for every positive value of

λ. That can be seen from Theorem 2.1 applied to the ideal generated by the first order derivatives of q_λ, whose part c is satisfied. In the following we will use the notion of limit set as defined below. The set L given by

$$L = \{x \in \mathbf{R}^n \mid \forall \varepsilon > 0\ \exists\ \lambda_\varepsilon\ s.t.\ \forall\lambda,\ 0 < \lambda < \lambda_\varepsilon,\ X_\lambda \cap B(x,\varepsilon) \neq \emptyset\}$$

is called the limit set of X_λ. For a multi-valued function with branches, by definition, the limit set will be simply the set of limits on the branches, assuming they exist.

THEOREM 3.3. *The following statements are true:*
(i) $\lim_{\lambda\downarrow 0} \min_{x\in\mathbf{R}^n} q_\lambda(x) = \inf_{x\in\mathbf{R}^n} p(x)$.
(ii) $\lim_{\lambda\downarrow 0} p(x_\lambda) = \inf_{x\in\mathbf{R}^n} p(x),\ \forall x_\lambda \in X_\lambda$.
(iii) *If the polynomial p has a minimum then* $L \subseteq \{x \in \mathbf{R}^n \mid p(x) = \min_{x\in\mathbf{R}^n} p(x)\}$.

Proof. (i) We consider two cases. First, we treat the case when p has a minimum attained at some point $\underline{x}$. Then

$$p(\underline{x}) = \inf_{x\in\mathbf{R}^n} p(x) \leqslant \inf_{x\in\mathbf{R}^n} (p(x) + \lambda\|x\|^{2m}) \leqslant p(\underline{x}) + \lambda\|\underline{x}\|^{2m}.$$

The above relation holds for every $\lambda > 0$, hence the relation is also valid at the limit $\lambda \downarrow 0$:

$$p(\underline{x}) \leqslant \lim_{\lambda\downarrow 0} \inf_{x\in\mathbf{R}^n} q_\lambda(x) \leqslant p(\underline{x})$$

which proves our statement.

Suppose now that $\inf_{x\in\mathbf{R}^n} p(x) = p_{inf}$, where p_{inf} may be finite or infinite. Let M be a real number $M > p_{inf}$, arbitrarily close to p_{inf}. Although p does not reach p_{inf}, there exists an $\underline{x} \neq 0$ such that $p(\underline{x}) < M$; then there is a $\underline{\varepsilon} > 0$ such that $p(\underline{x}) + \underline{\varepsilon} < M$. Define $\lambda_{\underline{\varepsilon}} = \underline{\varepsilon}/\|\underline{x}\|^{2m}$, where $\|x\|$ is the Minkowski norm. Then we have that for every $\lambda < \lambda_{\underline{\varepsilon}}$

$$\min_{x\in\mathbf{R}^n}[p(x) + \lambda\|x\|^{2m}] \leqslant p(\underline{x}) + \lambda\|\underline{x}\|^{2m} < M.$$

Since for every positive λ_1, λ_2 with $\lambda_1 < \lambda_2$ we have $q_{\lambda_1}(x) \leqslant q_{\lambda_2}(x),\ \forall x \in \mathbf{R}^n$, the limit exists and

$$\inf_{x\in R^n} p(x) \leqslant \lim_{\lambda\downarrow 0}[\min_{x\in R^n}[p(x) + \lambda\|x\|^{2m}]] \leqslant M$$

As M is arbitrarily close to p_{inf},

$$\lim_{\lambda\downarrow 0}[\min_{x\in R^n}[p(x) + \lambda\|x\|^{2m}]] = p_{inf}$$

(ii) It follows immediately from (i) since $\inf_{x\in\mathbf{R}^n} p(x) \leqslant p(x_\lambda) \leqslant q_\lambda(x_\lambda),\ \forall x_\lambda \in X_\lambda$.

(iii) Define $S = \{x \in \mathbf{R}^n \mid p(x) = \min_{x\in\mathbf{R}^n} p(x)\}$. We want to show $L \subseteq S$. Suppose $\exists x_0 \in L,\ x_0 \notin S$. Clearly $x_0 \notin S$ is equivalent to $p(x_0) \neq \min_{x\in\mathbf{R}^n} p(x)$. From the definition of the limit set L, we can construct a function which associates to every $\lambda > 0$ an $x_\lambda \in X_\lambda$ such that

$$\forall\varepsilon > 0\ \exists\lambda_\varepsilon > 0,\ s.t.\ \forall\lambda,\ 0 < \lambda < \lambda_\varepsilon \quad x_\lambda \in B(x_0, \varepsilon)$$

But this says exactly that $\lim_{\lambda\downarrow 0} x_\lambda = x_0$. As p is a continuous function we have that $\lim_{\lambda\downarrow 0} p(x_\lambda) = p(x_0)$. From part (ii) we have $\lim_{\lambda\downarrow 0} p(x_\lambda) = \min_{x\in\mathbf{R}^n} p(x)$. This is however in contradiction with our assumption that $p(x_0) \neq \min_{x\in\mathbf{R}^n} p(x)$. □

According to the theorem, one can obtain the infimum of p from the minima of the family of polynomials q_λ and, in case the minimum exists, can also obtain a set of points at which the minimum is attained, that is the limit set denoted here by L. To complete the discussion, we need to prove that L is a nonempty set, whenever the minimum of p is attained, and moreover is finite.

PROPOSITION 3.4. *The set L is finite.*

Proof. According to Theorem 2.10 of Cox et al. (1998), the number of critical points of q_λ is bounded by N, the dimension of the quotient ring, for every positive λ. It follows that the cardinality of X_λ is also bounded by N for every positive λ, since every point in X_λ is a critical point of q_λ. We will show that L has at most N points. Suppose that L has more than N distinct points and consider $N+1$ of them $l_1, \ldots, l_{N+1}$. Let $\delta > 0$ denote the smallest distance between any two of these points. For every $i = 1, \ldots, N+1$ construct the pairwise disjoint balls $B(l_i, \delta/2)$. By definition of L we have that there exists a $\lambda_{\delta/2} > 0$ such that every $B(l_i, \delta/2)$ has a nonempty intersection with X_λ, for each $\lambda \in (0, \lambda_{\delta/2})$. But for every $\lambda > 0$, X_λ has at most N elements, hence for each $\lambda \in (0, \lambda_{\delta/2})$, each of the $N+1$ disjoint balls should contain at least one of the N elements, which is impossible. Therefore L has at most N points. □

For our purposes, the non-emptiness is the most interesting part. In this way we have a guarantee that at least one point of global minimum is obtained with our procedure, provided the minimum is attained.

PROPOSITION 3.5. *If the polynomial p has a minimum, then L is nonempty.*

The proof of this proposition is given in the next section.

So far we have shown that with this method we can find the minimum value of every polynomial and some of the points in which the minimum is attained. In general we do not find all such critical points, especially when their number is infinite. One may wonder then which points we *do* find and the answer is partially given in the next proposition.

PROPOSITION 3.6. *If p has a minimum, then the set L contains only points of minimum of p which have minimal Minkowski norm. In other words, $L \subseteq \{x_0 \mid \|x_0\| = \min_{\{x \mid p(x)=p_{min}\}} \|x\|\}$ where p_{min} denotes the minimal value of p.*

Proof. (i) Let x_* be a point where the minimum of p is attained, of minimal Minkowski norm. We prove that

$$\|x_\lambda\| \leqslant \|x_*\|, \ \forall\lambda > 0, \ \forall \, x_\lambda \in X_\lambda.$$

From

$$q_\lambda(x_\lambda) = p(x_\lambda) + \lambda\|x_\lambda\|^{2m}, \ q_\lambda(x_*) = p(x_*) + \lambda \, \|x_*\|^{2m}$$

and $q_\lambda(x_\lambda) \leqslant q_\lambda(x_*)$ we have

$$\lambda \, [\|x_\lambda\|^{2m} - \|x_*\|^{2m}] \ \leqslant \ p(x_*) - p(x_\lambda) \ \leqslant \ 0$$

and therefore $\|x_\lambda\| \ \leqslant \ \|x_*\|, \ \forall\lambda > 0$.

(ii) Since p has a minimum, L is non-empty. As the norm is a continuous function, using the result of part i) we have

$$\forall x \in L, \quad \|x\| = \|\lim_{\lambda\downarrow 0} x_\lambda\| \leqslant \|x_*\|$$

But $\forall x \ \in \ L$ we have from Theorem 3.3, part (iii), and from the fact that x_* is a point of minimum of p of minimal Minkowski norm that $\|x\| \ \geqslant \ \|x_*\|$. Hence $\|x\| \ = \ \|x_*\|$ which implies $x \ \in \ \{x_0 \mid \|x_0\| \ = \ \min_{\{x \ | \ p(x)=p_{min}\}} \|x\|\}$ for every $x \in L$, so $L \subseteq \{x_0 \mid \|x_0\| = \min_{\{x \ | \ p(x)=p_{min}\}} \|x\|\}$.

□

Denote by X the multi-valued function defined on $(0, \lambda_1)$ which associates to each $\lambda \in (0, \lambda_1)$ the set X_λ. To give more insight into the properties of the branches of X, we prove their monotonicity. However, this result will not be used in the remainder of the paper.

PROPOSITION 3.7. *The multi-valued function X satisfies:* $\forall\lambda_1, \lambda_2$ *with* $0 < \lambda_1 < \lambda_2$ *and* $\forall x_{\lambda_1} \in X_{\lambda_1}, x'_{\lambda_2} \in X_{\lambda_2}$ *we have*

$$\|x_{\lambda_1}\| \geqslant \|x'_{\lambda_2}\|.$$

In particular, for one branch ($x = x'$) the proposition states that the branch is monotonously decreasing with respect to λ in Minkowski norm.

Proof. Given $\lambda_1 < \lambda_2$ we have

$$\begin{cases} q_{\lambda_1}(x_{\lambda_1}) \leqslant q_{\lambda_1}(x'_{\lambda_2}) \\ q_{\lambda_2}(x'_{\lambda_2}) \leqslant q_{\lambda_2}(x_{\lambda_1}) \end{cases}$$

or equivalently

$$\begin{cases} p(x_{\lambda_1}) + \lambda_1\|x_{\lambda_1}\|^{2m} - p(x'_{\lambda_2}) - \lambda_1\|x'_{\lambda_2}\|^{2m} \leqslant 0 \\ p(x'_{\lambda_2}) + \lambda_2\|x'_{\lambda_2}\|^{2m} - p(x_{\lambda_1}) - \lambda_2\|x_{\lambda_1}\|^{2m} \leqslant 0 \end{cases}$$

By adding the two inequalities we obtain

$$(\lambda_1 - \lambda_2)(\|x_{\lambda_1}\|^{2m} - \|x'_{\lambda_2}\|^{2m}) \leqslant 0$$

which implies $\|x_{\lambda_1}\| \geqslant \|x'_{\lambda_2}\|$ □

To summarize, we have constructed a family of polynomials q_λ, such that the infimum of our initial polynomial p can be obtained from the minima of the polynomials in the family, by letting the parameter λ decrease to 0. If the original polynomial has a minimum, the method will find at least one point in which the minimum is attained. We also have the Stetter-Möller method for solving the system given by the first order conditions, which is by construction a reduced Gröbner basis. Hence, we need to compute the limits of the eigenvalues of a matrix A_{q_λ} associated to the polynomial q_λ for λ going to 0.

In the following section, we propose a method for computing these limits.

4. Computing the minimum

From the previous section we know that we can find the minimum of the original polynomial p by computing the limits when λ goes to 0 of the eigenvalues of the matrix A_{q_λ}.

PROPOSITION 4.1. *For each polynomial $g \in \mathbf{C}[x_1, \ldots, x_n]$, the associated matrix A_g is a polynomial matrix in $1/\lambda$. In particular, for each $i = 1, \ldots, n$, A_{x_i} is polynomial in $1/\lambda$ and A_{q_λ} is polynomial in $1/\lambda$.*

Proof. The proof goes by induction on the number of reduction steps. Recall that our Gröbner basis has a particular form in which the leading monomials are pure powers of the variables and λ appears only in the leading coefficient. Hence we start with constant entries but, due to the particular form of the Gröbner basis, whenever we make a reduction step (see for example Cox et al., 1998), we introduce a $1/\lambda$ or a power of it in some entries. Therefore, the entries of the final matrix will be polynomials in $1/\lambda$. □

In order to underline the dependency on λ, we denote $A_g = A_g(\lambda)$, where g is an arbitrary polynomial. The size of $A_g(\lambda)$ is given by the dimension of the basis B which is $N = (2m - 1)^n$.

Recall the interpretation of the eigenvalues in the Stetter-Möller method. The eigenvalues of $A_g(\lambda)$ are the values of the polynomial g evaluated at the critical points of q_λ. In particular, when $g = q_\lambda$, these eigenvalues are the critical values of q_λ. The global minimal value of q_λ is among them and it converges to the infimum of p when $\lambda \downarrow 0$. The eigenvectors of A_{q_λ} will give the corresponding points and their limits for $\lambda \downarrow 0$ will allow us to read off a critical point of p where the minimum is attained. However if the set of critical points of p is not finite we are not able in general to find the whole set, but we find a finite subset of it.

For this reason, we study in the following the limits for λ decreasing to 0 of the eigenvalues of a matrix $A_g(\lambda)$. The equation

$$\det(A_g(\lambda) - zI) = 0, \quad \lambda > 0, \quad z \in \mathbf{C}$$

is satisfied if and only if

$$\lambda^k \det(A_g(\lambda) - zI) = 0, \quad \lambda > 0, \quad z \in \mathbf{C} \tag{2}$$

where k is the highest power of $1/\lambda$ appearing in the determinant. The second equation, polynomial in both z and λ, was studied extensively in the literature. Its solutions $z(\lambda)$ which satisfy the equation for every positive λ are known as *algebraic functions* (see Bliss, 1933). An algebraic function is a multi-valued function having a finite number of branches $\zeta_i(\lambda),\ i = 1, \dots N$. The values of each branch around an arbitrary $\lambda_0 \geqslant 0$ are given by a Puiseux expansion in rational powers of $\lambda - \lambda_0$. To be more precise, the following proposition holds (Bliss, 1933, Theorem 13.1).

PROPOSITION 4.2. *In a neighborhood V of every finite point $\lambda = \lambda_0$ all values of an algebraic function $z(\lambda)$ are determined by branches of the form*

$$\lambda = \lambda_0 + t^r, \qquad z = z_{-\kappa} t^{-\kappa} + z_{-\kappa+1} t^{-\kappa+1} + \dots + z_0 + z_1 t + \dots \tag{3}$$

in which r is a positive integer, the coefficients $z_{-\kappa}, z_{-\kappa+1}, \dots$ indicated are complex, possibly zero, and κ is a non-negative integer. For a value $\lambda \neq \lambda_0$ in V, (3) determines r distinct values of $z(\lambda)$ when the r values of the root $t = (\lambda - \lambda_0)^{1/r}$ are substituted in the series for z.

We are now able to give the proof of a previously stated proposition.

Proof of Proposition 3.5. In the definition of L, X_λ denotes the set of real points where the minimum of q_λ is attained. To show that L is nonempty we first prove that X_λ is continuous on branches on an interval $(0, \lambda_0)$ for λ_0 sufficiently small. For that, we refer to Stetter-Möller theory. It is known that the coordinates of the point in $\mathbf{R}^n$ where the minimum of q_λ is attained, i.e. the coordinates of X_λ, can be obtained as the eigenvalues of the matrices $A_{x_i}(\lambda)$ for $i = 1, \dots, n$, where $A_{x_i}(\lambda)$ denotes the linear mapping associated to the polynomial x_i (see Section 2).

From Proposition 4.1 we have that the matrices $A_{x_i}(\lambda)$ are polynomial matrices in $1/\lambda$. So, the eigenvalues of $A_{x_i}(\lambda)$ are the solutions of the equation in z, $\det(A_{x_i}(\lambda) - zI) = 0$ or equivalently, $\lambda^k \det(A_{x_i}(\lambda) - zI) = 0$ where k is the highest power of $1/\lambda$ appearing in the determinant. As the equation is polynomial in z and λ, the solutions $X_i(\lambda)$ are algebraic functions. For every fixed positive λ the algebraic functions admit in a neighborhood of λ an expansion in which radicals or (a finite number of) terms with negative exponent may be involved (see Proposition 4.2). This implies in particular that the branches of X_i as functions of λ are continuous in a right neighborhood of $\lambda = 0$. Since $X_i(\lambda)$ are coordinates of X_λ, then also X_λ is continuous in a right neighborhood.

Next we argue that when p has a minimum, there will be a branch of X_λ which does not contain negative powers of λ in its expansion around 0. As p has a minimum, there exists a point in which the minimum is attained. We know that the branches of X_λ are bounded in the Minkowski norm by such a point (see

Proposition 3.6, first part of the proof). Hence X_λ will have finite limits on the branches when $\lambda \downarrow 0$ and all these limits belong to the limit set L which is therefore nonempty. □

Recall that we want to compute the limits of the branches when $\lambda \downarrow 0$ so in our case $\lambda_0 = 0$ and V is a neighborhood of 0. The expansion of a branch of an algebraic function may have a finite number of terms containing negative powers of λ. We say that a branch has an *infinite limit* when $\lambda \downarrow 0$ if its expansion contains negative powers of λ. Otherwise we say that it has *finite limit*. The branches that have finite limits will tend, when $\lambda \downarrow 0$, to z_0, the term of the expansion which does not depend on λ (see (3)).

Let

$$\det(A_g(\lambda) - zI) = f(\lambda, z) = 1/\lambda^k f_0(z) + 1/\lambda^{k-1} f_1(z) + \ldots + f_k(z).$$

where $f_0, f_1, \ldots, f_k$ are polynomials in z. Then Equation (2) becomes

$$f_0(z) + \lambda f_1(z) + \ldots + \lambda^k f_k(z) = 0$$

We can easily see from Proposition 4.2 that the finite limits are solutions of the equation $f_0(z) = 0$. In fact one can show a bit more.

PROPOSITION 4.3. *The critical values of the polynomial q_λ define a finite number of branches having, when $\lambda \downarrow 0$, finite or infinite limits. The set of finite limits of q_λ coincides with the set of solutions of $f_0(z) = 0$.*

Proof. The first part of the theorem was already discussed. For the last part, consider $\zeta(\lambda)$ a branch having a finite limit. By replacing $\zeta(\lambda)$ by its expansion, one can easily see that the λ-free term in the expansion, is a solution of $f_0(z) = 0$. Hence the number of branches having a finite limit is at most equal to the degree of f_0, denoted by d. We will show that in fact the equality holds, hence the two sets must be equal. For this purpose we consider next the branches having infinite limits, i.e. their expansion contains negative powers of λ. Let $\zeta(\lambda)$ be a solution of (2) whose expansion contains negative powers of λ. Then $\omega(\lambda) = 1/\zeta(\lambda)$ is a solution of the equation $f(\lambda, 1/w) = 0$ or equivalently

$$w^N f(\lambda, 1/w) = 0. \tag{4}$$

Note that the second equation was obtained by bringing the terms in $f(\lambda, 1/w)$ to the common denominator w^N and taking afterwards the numerator equal to 0. Remark that $\lim_{\lambda\downarrow 0} \omega(\lambda) = 0$ as can be seen for example from the expansion of $\zeta(\lambda)$. Hence $\omega(\lambda)$ is solution of the polynomial Equation (4) and, having limit 0, is a finite solution of the equation. Rewriting the Equation (4) we have

$$w^N[f_0(1/w) + \lambda f_1(1/w) + \ldots + \lambda^k f_k(1/w)] = 0$$

and we need to compute the number of branches $w(\lambda)$ that tend to 0 when $\lambda \downarrow 0$. But as we have argued before, every 0 limit of a branch of $w(\lambda)$ is a root of the

λ-free term, $w^N f_0(1/w)$. But $w^N f_0(1/w)$ has exactly $N-d$ zero roots, where d was the degree of f_0. Hence the number of branches of $w(\lambda)$ having the limit 0, which equals the number of branches of $z(\lambda)$ having infinite limits, is at most $N-d$. To conclude, we have exactly N branches having either finite or infinite limit and we have shown that among them at most d have finite limits and at most $N-d$ have infinite limits. Hence there must be *exactly* d branches having finite limits and *exactly* $N-d$ having infinite limits. □

PROPOSITION 4.4. *Suppose that p has a minimum and $\underline{x}$ is an isolated point of minimum of the polynomial p. There exists a branch x_λ of (local) minima of q_λ convergent to $\underline{x}$ for $\lambda \downarrow 0$.*

Proof. As p is a polynomial and $\underline{x}$ is an isolated point of (global) minimum, there exists a convex neighborhood V of $\underline{x}$, where p is strictly convex. The function $x_1^{2m} + x_2^{2m} + \ldots + x_n^{2m}$ is strictly convex on R^n. It follows immediately that for every $\lambda > 0$, q_λ is strictly convex in V. Let $\varepsilon > 0$ such that $\overline{B(\underline{x}, \varepsilon)} \subset V$. Next, we show that the unique point of minimum of q_λ on $\overline{B(\underline{x}, \varepsilon)}$ is, for every λ sufficiently small, a point of $B(\underline{x}, \varepsilon)$. Remark that, because p is strictly convex in V, $\min_{x\in\partial B(\underline{x},\varepsilon)} p(x) > \min_{x\in B(\underline{x},\varepsilon)} p(x) = p(\underline{x})$. From this inequality and the fact that $\lim_{\lambda\downarrow 0} q_\lambda(\underline{x}) = p(\underline{x})$, we have

$$\exists\lambda_\varepsilon > 0 \text{ such that } \forall\lambda, 0<\lambda<\lambda_\varepsilon, \quad q_\lambda(\underline{x}) < \min_{x\in\partial B(\underline{x},\varepsilon)} p(x) \leqslant \min_{x\in\partial B(\underline{x},\varepsilon)} q_\lambda(x).$$

That implies that the minimum of q_λ does not lie on the border. Hence, for every $\lambda < \lambda_\varepsilon$ we have a unique $x_\lambda \in B(\underline{x}, \varepsilon)$ such that $q_\lambda(x_\lambda) = \min_{x\in B(\underline{x},\varepsilon)} q_\lambda(x)$. In other words, x_λ is a local minimum of q_λ and $(x_\lambda)_{\lambda>0}$ a branch of local minima of q_λ, convergent to $\underline{x}$ for $\lambda \downarrow 0$. □

COROLLARY 4.5. *If p has a minimum, then for each isolated point of (global) minimum of the polynomial p there exists a branch of (local) minima of q_λ convergent to it for $\lambda \downarrow 0$. In particular, if p has a finite number of points of minimum, they are all limits of branches of (local) minima of q_λ.*

Proof. For each isolated point of minimum of p we apply Proposition 4.4. For the second part, remark that if p has a finite number of points of minimum, they are all isolated. □

THEOREM 4.6. *If p has a minimum then the set $p^{-1}(\{\min_{x\in\mathbf{R}^n} p(x)\})$ consists of one or more connected components. In each component there exists at least one point which is the limit of a branch of (local) minima of q_λ when $\lambda \downarrow 0$. Moreover, these points have minimal Minkowski norm inside the component.*

Proof. Note that the number of connected components of $p^{-1}(\{p_{min}\})$ is finite (see Bochnak et al., 1987, Th 2.4.5), where $p_{min} = \min_{x\in\mathbf{R}^n} p(x)$. Pick a point, say

$x(j)$, in each component C_j, where

$$C = \bigcup_{j \in J} C_j = \{x \in \mathbf{R}^n \mid p(x) = p_{min}\}.$$

Let $M_j = \|x(j)\|$ and $M > \max_{j \in J} M_j$.

We want to show that for every $j \in J$, there will be a local minimum of q_λ whose points of minimum are in the Minkowski ball $B(0, M)$ and converge to an element of C_j. If this holds then, from the local minima of q_λ, we obtain at least one point in each component C_j.

Note that in each component C_j there is a point, namely $x(j)$, such that

$$q_\lambda(x(j)) < p_{min} + \lambda M^{2m} \leqslant q_\lambda(x),\ \forall x \notin B(0, M).$$

Hence

$$q_\lambda(x(j)) < q_\lambda(x),\ \forall x \notin B(0, M)$$

and the minima of q_λ corresponding to every component C_j, provided they exist, are in the Minkowski ball $B(0, M)$.

Consider $q_\lambda \big|_{\overline{B(0,M)}}$. The number of connected components of $C \bigcap \overline{B(0, M)}$ is still finite since the set $\{x \in \mathbf{R}^n \mid \|x\|^{2m} \leqslant M^{2m}, p(x) = p_{min}\}$ is a semi-algebraic set (Bochnak et al., 1987, Th 2.4.5). Denote them by D_l. Since $\overline{B(0, M)}$ is a compact set and the sets D_l are disjoint and closed, it follows that $\exists \varepsilon_0 > 0$ such that $\forall l_1 \neq l_2\ d(D_{l_1}, D_{l_2}) > \varepsilon_0$, where d denotes the Minkowski distance between sets.

Define the neighborhood of a component D_l as

$$N_{\varepsilon_0/3}(D_l) = \{x \in B(0, M) |\ d(x, D_l) < \varepsilon_0/3\}.$$

We want to show that the minimum of $q_\lambda \Big|_{\overline{N_{\varepsilon_0/3}(D_l)}}$ is not attained on the border of $\overline{N_{\varepsilon_0/3}(D_l)}$. Note that any point on the border satisfies either $\|x\| = M$ or $d(x, D_l) = \varepsilon_0/3$. We already know that the points on the border of $B(0, M)$ are not minima.

Let $\bar{p} = \min_{\bigcup_l (\partial N_{\varepsilon_0/3}(D_l) \bigcap B(0,M))} p(x)$. Then $\bar{p} > p_{min}$. We have $q_\lambda \Big|_{\partial N_{\varepsilon_0/3}(D_l) \cap B(0,M)} \geqslant \bar{p}$.

On the other hand, for any $x \in D_l$ we have $q_\lambda(x) = p_{min} + \lambda \|x\|^{2m} \leqslant p_{min} + \lambda M^{2m} < \bar{p}$ for λ sufficiently small, namely $\lambda < (\bar{p} - p_{min})/M^{2m}$. Therefore, if $\lambda < (\bar{p} - p_{min})/M^{2m}$ then $\min_{x \in \overline{N_{\varepsilon_0/3}(D_l)}} q_\lambda(x)$ is attained in the open set, not on the boundary.

We have proved that for λ smaller than a certain value, for every component D_l there exists an open neighborhood of it containing points of local minimum of $q_\lambda \big|_{\overline{B(0,M)}}$.

Let x_λ^l be a global minimizer of $q_\lambda \Big|_{\overline{N_{\varepsilon_0/3}(D_l)}}$. Then x_λ^l is a local minimizer of q_λ (on R^n). Since x_λ^l is local minimizer, it is convergent, as in the proof of Proposition 3.5, to a point, say $x_* \in \overline{N_{\varepsilon_0/3}(D_l)}$.

We want to show that $x_* \in D_l$. We have $p(x) \leqslant q_\lambda(x)$ and $\lim_{\lambda\downarrow 0} q_\lambda(x) = p(x), \ \forall x \in R^n$. Hence $p(x^l_\lambda) \leqslant q_\lambda(x^l_\lambda) \leqslant q_\lambda(x_*)$. When $\lambda \downarrow 0$ we obtain $\lim_{\lambda\downarrow 0} q_\lambda(x^l_\lambda) = p(x_*)$.

Take $x_0 \in D_l$. We have $q_\lambda(x^l_\lambda) \leqslant q_\lambda(x_0)$ and at the limit it becomes $p(x_*) \leqslant p_{min}$ or in fact $p(x_*) = p_{min}$. This implies that $x_* \in D_l$. □

We have proved here (Theorem 4.6) that, if p has a minimum, any algorithm which is able to compute all the limits of the branches of (local) minima of q_λ, computes in fact at least one point in each connected component of the set of minimal values of the polynomial p. Such an algorithm is described in the following section (Algorithm 4.10).

4.1. CASE: THE POLYNOMIAL p HAS A MINIMUM

From Theorem 3.3 we know that $\min_{x\in R^n} q_\lambda(x) = q_\lambda(x_\lambda)$ converges to $\min_{x\in R^n} p(x)$. But $q_\lambda(x_\lambda)$ satisfies Equation (2), so it is a branch of the algebraic function associated to Equation (2), for $g = q_\lambda$. Moreover, we know it has a finite limit. Hence $\lim_{\lambda\downarrow 0} q_\lambda(x_\lambda)$ will be a root of f_0. The smallest real root is our candidate for the minimum of p. Note that we have been working over the field of complex numbers and it is possible that the smallest real root is a value of p attained in a complex point. Hence, before deciding that the smallest real root is the minimum of p, we need to do a check at the point where the minimum is attained. We will discuss this issue later, but until then, in order to make the discussion easier, we will assume that the smallest real eigenvalue is indeed the minimum.

The way to compute $\min_{x\in R^n} p(x)$ becomes more clear now. Having constructed the matrix A_{q_λ}, one can calculate $\det(A_{q_\lambda} - zI)$, polynomial in $1/\lambda$ and z, then isolate the coefficient of the largest power of $1/\lambda$. This is a polynomial in z whose smallest real root gives us the minimum of p.

We have now a straightforward way to compute the minimum of our polynomial p. However, the drawback of using the determinant is that, besides the high computational complexity, it will not tell us anything about the corresponding eigenvectors. As we already remarked, knowing the eigenvectors may be helpful in finding not only the minimum but also (at least) a point in which the minimum is attained. Hence we need a more “sophisticated” method for the actual calculations.

We describe here a method for computing the finite limits of the eigenvalues, without actually computing the determinant. It will be clear that with this new method, we can not only find the corresponding eigenvectors but also we do less calculations, as we only need one term of the determinant.

The method is a special case of the well-known algorithm of Forney, (1975) for minimizing the sum of the row degrees of a polynomial matrix over an equivalence class of polynomial matrices. With this method we obtain the coefficient of the highest power of $1/\lambda$ in the expression of the determinant $\det(A_g(\lambda) - zI)$ as a polynomial matrix in z. This coefficient is a matrix, polynomial in z and independent of λ. After applying linearization techniques (see Gohberg et al., 1982, § 7.2) we reduce it to the problem of finding the eigenvalues of a pencil. Since the original matrix is nonsingular over $\mathbf{R}[z]$ and the linearization procedure leaves the determinant unchanged, the generalized eigenvalue problem obtained is always nonsingular.

Remark that the problem of finding the minimum of a polynomial and some point where this is attained is reduced to solving a generalized eigenvalue problem. For this new problem, a large variety of algorithms exists and they can handle quite large matrices.

Let us describe now in more detail how to find the coefficient of the highest power of $1/\lambda$ in the expression of the determinant $\det(A_g(\lambda) - zI)$. The procedure is quite general and can be applied to an arbitrary polynomial matrix. Let $B(\mu)$ be a polynomial matrix in μ. The degree of the i-th row, denoted d_i, is the highest degree in μ of all its entries. The total row degree of the matrix is the sum of its row degrees. The associated high-order coefficient matrix, denoted HOCM, is constructed by retaining from each entry of the i-th row, the coefficient of μ^{d_i}.

The algorithm for finding the leading term of $\det(B(\mu))$, i.e. the term containing the highest power of μ in the expression of the determinant $\det(B(\mu))$, is based on the following:

PROPOSITION 4.7. *Let $B(\mu)$ be a polynomial matrix in μ and let d denote its total row degree. The leading term of the polynomial* $\det(B(\mu))$ *in μ is*
$\det(\mathrm{HOCM}(B(\mu)))\mu^d$
if and only if $\mathrm{HOCM}(B(\mu))$ *is nonsingular.*

Proof. It follows immediately from the well-known formula for computing determinants. □

If we apply the above procedure for $\mu = 1/\lambda$, we can find the leading coefficient of $\det(A_g(\lambda) - zI)$, polynomial matrix in $1/\lambda$, for any polynomial g. Note that by construction the total row degree of $(A_g(\mu) - zI)^T$ is in general much smaller than the total row degree of $A_g(\mu) - zI$. Therefore, for computational reasons, we work with $(A_g(\mu) - zI)^T$.

ALGORITHM 4.8. *The following procedure returns a matrix, polynomial in μ and rational in z, of minimal total row degree in μ, equivalent to the input matrix $(A_g(\mu) - zI)^T$.*

1. *Input: $B(\mu) \leftarrow (A_g(\mu) - zI)^T, \Delta \leftarrow 1$*
2. *Compute $d_i, i = 1, \dots, N$ and* $\mathrm{HOCM}(B(\mu))$. *If* $\mathrm{HOCM}(B(\mu))$ *is nonsingular, then go to Step 7.*
3. *Else compute a nonzero vector $v = (v_1, \dots, v_N)$ in the left kernel of* $\mathrm{HOCM}(B(\mu))$. *The vector can be chosen polynomial in z.*
4. *Construct the vector $\tilde{v} = (v_1\mu^{d_*-d_1}, \dots, v_N\mu^{d_*-d_N})$, where* $d_* = \max_{\{i=1,\dots,N \mid v_i \neq 0\}} d_i$.

5. *Construct a matrix* $L(\mu, z)$ *from the identity matrix by replacing its i-th row by* $\tilde{v}$*, where i is chosen such that* $d_i = d_*$.
6. $B(\mu) \leftarrow L(\mu, z)B(\mu)$, $\Delta \leftarrow \Delta \cdot \det(L(\mu, z))$. *Go to Step 2.*
7. *Output:* $\overline{A}g(\mu, z) \leftarrow B(\mu)$, *with* $\det((A_g(\mu) - zI)^T) = 1/\Delta \cdot \det(B(\mu))$ *and* $\mathrm{HOCM}(B(\mu))$ *nonsingular.*

As $A_g(\mu) - zI$ is nonsingular, i.e., its determinant is non-identically zero, the degree in μ in the expression $\det(A_g(\mu) - zI)$ is a positive natural number $\tilde{d}$. As we run the algorithm, the total row degree of the matrix is decreased by 1, at least, every time we execute step *6*. Hence the algorithm stops after a finite number of steps, when the total row degree of $B(\mu)$ reaches the value $\tilde{d}$.

Remark that $\mathrm{HOCM}(B(\mu))$ is polynomial matrix in z hence a vector as in step *3* always exists. Remark also that the determinant of $L(\mu, z)$ from step *5* does not depend on μ. It may depend on z, therefore we need the corrections Δ. Matrices like $L(\mu, z)$ depending on a parameter μ, whose determinant does not depend on μ are called *z*-modular or unimodular over $\mathbf{R}[z]$.

Since at step *6* we multiply with *z*-modular matrices, our HOCM may become polynomial, not linear, in z. The nonsingular polynomial matrix in z can be brought by a linearization procedure (see Gohberg et al., 1982, § 7.2) into an equivalent matrix, linear in z of a larger dimension. Note however that in the reduction process while multiplying on the left with z-modular matrices we introduce some new solutions. Hence we must keep track of the solutions we introduce and subtract them in the end.

To be more precise, after running the algorithm we have

$$R(\mu, z) = L(\mu, z)(A_g(\mu) - zI)^T ,$$

where $R(\mu, z)$ has a nonsingular HOCM and $L(\mu, z)$ is z-modular. For their determinants, the following holds:

$$\det(R(\mu, z)) = \det(L(\mu, z)) \det(A_g(\mu) - zI)$$

and using Proposition 4.7 and the fact that $\det(L(\mu, z))$, which equals our final value of Δ in the algorithm, does not depend on μ, it follows that the leading term of $\det(A_g(\mu) - zI)$ in μ satisfies

$$\mathrm{lt}(\det(A_g(\mu) - zI)) = (\det(L(\mu, z)))^{-1} \det(\mathrm{HOCM}(R(\mu, z))).$$

The roots of $\det(L(\mu, z))$ are artificially introduced so we must eliminate them.

The algorithm can be applied in general for finding a (left-)equivalent representation of a matrix of minimal total row degree. In the following we give a small example to illustrate how the algorithm works.

EXAMPLE 4.9. Consider a matrix $M(\mu)$, polynomial in μ, of non-minimal total row degree. $M(\mu)$ plays the role of $A_g(\mu)$, the difference being that $M(\mu)$ is not associated to a polynomial. Let

$$M(\mu) = \begin{pmatrix} \mu^2 & 0 & \mu \\ 1 & 0 & -2 \\ \mu^3 & \mu & \mu^2 \end{pmatrix}.$$

The matrix $B(\mu) = (M(\mu) - zI)^T$ becomes

$$B(\mu) = \begin{pmatrix} \mu^2 - z & 1 & \mu^3 \\ 0 & -z & \mu \\ \mu & -2 & \mu^2 - z \end{pmatrix}$$

with the row degree vector $(3, 1, 2)$, hence the total row degree 6. However its HOCM is singular,

$$\text{HOCM}(B(\mu)) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

hence its total row degree is not minimal. Pick up a vector in the left kernel of HOCM$(B(\mu))$, say $v = (-1, 1, 0)$ and construct $\tilde{v} = (-1, \mu^2, 0)$. The matrix $L(\mu, z)$ becomes

$$L(\mu, z) \leftarrow \begin{pmatrix} -1 & \mu^2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and by multiplication on the right with $B(\mu)$,

$$B(\mu) \leftarrow \begin{pmatrix} -\mu^2 + z & -1 - z\mu^2 & 0 \\ 0 & -z & \mu \\ \mu & -2 & \mu^2 - z \end{pmatrix} \text{ and } \Delta \leftarrow -1.$$

Since the new matrix has a singular HOCM, we return to step 2 and continue the reduction procedure. Hence

$$B(\mu) \leftarrow \begin{pmatrix} -\mu^2 + z & -1 - z\mu^2 & 0 \\ 0 & -z & \mu \\ \mu & \mu z - 2 & -z \end{pmatrix} \text{ and } \Delta \leftarrow -1.$$

Remark that another reduction step is necessary and finally we obtain

$$B(\mu) = \begin{pmatrix} z & -2\mu - 1 & -\mu z \\ 0 & -z & \mu \\ \mu & \mu z - 2 & -z \end{pmatrix}$$

whose HOCM is nonsingular. Remark that the determinant of this matrix is $-2\mu^2 z^2 + z^3 + 2z\mu - 2\mu^3 - \mu^2$ and it is equal to $\Delta \det(M(\mu) - zI)$. In this example, the total row degree was reduced from 6 to the minimal row degree which is 3.

In general, when Δ depends on z we introduce false solutions for $\det(A_g(\mu) - zI)$ during the reduction procedure. An improvement on the algorithm would be to avoid introducing such solutions or if we do, to eliminate them in a smarter way. The problem reduces basically to the following one: having a polynomial matrix $\tilde{M}(z)$ and a polynomial $\tilde{m}(z)$ which divides its determinant, find a polynomial matrix whose determinant is $\det \tilde{M}(z)/\tilde{m}(z)$. Obviously, such a matrix exists as well as an algorithm to compute it. The question is whether we can compute such a matrix in an efficient way.

Note that the eigenvectors of the matrix $\mathrm{HOCM}(R(1/\lambda, z))$, polynomial in z, preserve the property of the Stetter vectors. Namely, when the eigenspace is 1-dimensional, the eigenvector is the basis vector evaluated at the critical point. This follows from the fact that we multiply the matrix $(A_g(\lambda) - zI)^T$ only on the left-hand side, hence its Stetter eigenvectors are preserved. In the end we obtain,

$$L(1/\lambda, z)(A_g(\lambda) - zI)^T v_\lambda = 0, \ \forall\lambda > 0.$$

By premultiplying with $\mathtt{diag}(\lambda^{d_1}, \ldots, \lambda^{d_N})$, where d_j is the (minimal) row degree of row j we obtain an N-dimensional equation in λ, valid for every $\lambda > 0$ and well-defined in $\lambda = 0$. Then the equation must hold also for $\lambda = 0$, but that is exactly $\mathrm{HOCM}(R(1/\lambda, z)) \cdot \lim_{\lambda\downarrow 0} v_\lambda = 0$. That insures us that the eigenvectors of $\mathrm{HOCM}(R(1/\lambda, z))$ will indeed correspond to critical points of p.

In Cox et al. (1998) a method is proposed for choosing the polynomial g such that the left-eigenspaces of A_g are 1-dimensional, so that one can 'read' immediately not only the values of g on $V(I)$ but also the points where the value is obtained. As suggested there, g can be an arbitrary linear combination of the variables, i.e. $g = c_1x_1 + \ldots + c_nx_n$ where $c_1, \ldots, c_n$ are complex constants. Such choice may be important if one wants to use the properties of the Stetter vectors.

To resume, the computational procedure we suggest is:

ALGORITHM 4.10. *The following procedure can be used for computing the minimum of p.*

1. *Select a polynomial g and construct the corresponding matrix $A_g(\lambda)$.*
2. *Compute the* $\mathrm{HOCM}(\overline{A}g(1/\lambda), z))$ *by running the Algorithm 4.8.*
3. *Compute those values of z for which* $\mathrm{HOCM}(\overline{A}g(1/\lambda, z))$ *(polynomial matrix in z) becomes singular. Compute the corresponding eigenvectors.*
4. *Read off the critical points from the eigenvectors by using the Stetter interpretation.*
5. *Evaluate the polynomial p at all these critical points and identify the global minimum as the smallest value.*

The choice of the polynomial g at step *1* is left to the user. It may equal p or q_λ, or a linear combination of the variables which (ideally) leads to 1-dimensional eigenspaces and therefore allows an immediate reading of the critical points. Note however that for the latter choice of g, the assumption that the polynomial p has a minimum is essential! In case this is not true, one can find the value of the finite infimum (if this exists) only in a direct way, by choosing g equal to p or q_λ.

4.2. CASE: THE POLYNOMIAL p HAS AN INFIMUM

In the previous section we have described an algorithm for computing the global minimum of a polynomial, in case it exists. When the same procedure is applied for g equal p or q_λ, the algorithm actually computes the value of the (finite) infimum, if that exists. We believe this is one of the very important features of the algorithm.

At this point we do not have a direct way of deciding whether the infimum is finite or not. However, the following procedure can in principle be used to decide this. Compute the candidate for the finite infimum by running the Algorithm 4.10 . Let us denote the obtained value by c. Then form the polynomial $(p - c + \alpha)^2$, α being a positive constant, and run the algorithm again. If c was indeed the infimum of p, then the new polynomial must have infimum α^2. If there are values of p strictly smaller than c, then due to the continuity of p there must exist a point x such that $p(x) = c - \alpha$. Hence the new polynomial will have the minimum equal to 0.

Further research is useful into finding a direct way to decide upon this matter.

5. Example

We consider here rather small examples. There are a few reasons for our choices. The first one is that the method we have proposed requires a number of calculations that increases rapidly with the degree of the polynomial and the number of variables. The second, and more important reason, is that in these cases we already know the minimum and the set of points where it is attained, therefore it is possible to analyze the algorithm in these specific examples. We considered interesting the case of an infinite number of critical points. In the finite case we know from the theory that the algorithm finds all the points.

EXAMPLE 5.1. Let $p(x_1, x_2) = \left(x_1^2 + x_2^2 - 1\right)^2$. The minimum is obviously 0 and the set of points where it is attained is the circle of radius 1, centered in $(0, 0)$. We apply the algorithm by first constructing the family of polynomials

$$q_\lambda (x_1, x_2) = \left(x_1^2 + x_2^2 - 1\right)^2 + \lambda \left(x_1^6 + x_2^6\right).$$

The power in the extra-term was chosen to be an even number strictly larger than 4, the total degree of p. Next we construct our matrices using the Stetter-Möller method.

We follow the Algorithm 4.10. As g polynomial we choose $g = x_1 + 3x_2$. We construct the associated matrix $A_g(\lambda)$, polynomial in $1/\lambda$, which is square of size $(6 - 1)^2 = 25$. The total row degree of $(A_g(\lambda) - zI)^T$ is 12. However it is not minimal, i.e., the highest power of $1/\lambda$ appearing in the determinant of $A_g(1/\lambda, z) = (A_g(\lambda) - zI)^T$ is actually 6 as results by running the total row degree reduction algorithm of Forney (Algorithm 4.8) on $A_g(1/\lambda, z)$ which will return the matrix $\bar{A}_g(1/\lambda, z)$. At this point we have also obtained the coefficient of the highest power of $1/\lambda$ in the expression $\det(A_g(1/\lambda, z))$. This is the determinant of the HOCM of $\bar{A}_g(1/\lambda, z)$. Computing the eigenvalues of HOCM, i.e. the zeroes of the determinant of HOCM, we obtain

$$\left\{0, 1, -1, 3, -3, 2\sqrt{2}, -2\sqrt{2}, \sqrt{2}, -\sqrt{2}\right\}.$$

All eigenvalues have multiplicity 1, therefore from the corresponding eigenvectors we read off the following corresponding points:

$$\Big\{(0,0), (1,0), (-1,0), (0,1), (0,-1), \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right), \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)\Big\}.$$

Evaluating the polynomial p at these points, we conclude that the candidate for the minimum is 0 and it is attained at all points above except $(0, 0)$. To be completely safe, we should check that p has indeed a minimum. It is easy to check that p does not have a finite infimum and we do that by rerunning the algorithm for $g = p$. The value returned by the algorithm equals 0, the minimal value we have found already.

In order to check that the polynomial does not have an infinite infimum, we need to apply the trick described in Section 4.2. Therefore we run again the algorithm for $g = (p + 1)^2$ and obtain that the minimum of the new polynomial equals 1. The critical points of the new polynomial coincide with the critical points of p. If p had an infinite infimum, $(p+1)^2$ should have had a minimum at 0. Therefore we conclude that the minimum of p is indeed 0.

Remark that the values $(\pm\sqrt{2}/2, \pm\sqrt{2}/2)$ are points where the minimum of p is attained, of minimal Minkowski norm. This was predicted in Proposition 3.6. However we obtain some extra points which in this case are points of maximal Minkowski norm. It is an open question whether we find points of maximal Minkowski norm in every connected component whenever the component is bounded.

EXAMPLE 5.2. Let us consider now a polynomial having a finite infimum, as in Uteshev and Cherkasov (1998):

$$p(x_1, x_2) = x_1^2 x_2^4 + x_1 x_2^2 + x_1^2.$$

We run the algorithm for $g = p$ and obtain the results 0 (with multiplicity 3) and $-1/4$ (with multiplicity 12). Obviously, the candidate for the infimum is $-1/4$, being the smallest among the two. If $-1/4$ were a minimum of the polynomial, then we should be able to find out the coordinates of the respective point by rerunning the algorithm for $g = x_1$ and $g = x_2$. But by doing so, we only obtain the point $(0, 0)$, hence we conclude that $-1/4$ is not attained. We should still check whether the infimum is not $-\infty$ as we did in the first example, however we do not make the computations here.

6. Conclusions

The proposed method is guaranteed to find the global minimum of a general polynomial. Moreover, if the minimum does not exist, we can decide if the infimum is finite or not, and give its value in the first case. To the best of our knowledge this problem did not receive until now a solution in the general case.

The approach translates the original problem into a generalized eigenvalue problem. This may open up the possibility for numerical calculations.

Another very important feature of the algorithm is that it returns a point in every connected component of the set of (global) minimizers. Using the algorithm we can in fact answer a different problem as well. Given a set of polynomial equations $f_i(x_1, \dots, x_n) = 0,\ i = 1, \dots s$, we can find a point in every connected component of the solution set, simply by minimizing $f = \sum_{i=1}^{s} f_i^2$. Such problems received a lot of attention (see Basu et al. (1998) and the references contained therein).

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